On The Derivation and Consistent Use of Growth and Discount Rates For Future Earnings

Gary R. Albrecht and John C. Moorhouse*

In litigation involving personal injury, wrongful death, job discrimination, and breach of employment contracts, questions concerning the estimates of future earnings and the choice of an appropriate discount rate arise. We address the question of whether, in order to estimate the growth of future earnings and then to discount future earnings, it is necessary to forecast the inflation rate. Beginning with the neoclassical theory of the firm and intertemporal utility maximization theory we derive results which show that it is not necessary to forecast the rate of inflation; if discrete growth rates are combined with a discrete growth model or, if continuous growth rates are combined with a continuous growth model, the inflation rate cancels out. Our results obviate the need to forecast the inflation rate.

I. Introduction

In litigation involving personal injury, wrongful death, job discrimination, and breach of employment contracts, expert witnesses often are called upon to calculate the present discounted value (PDV) of an estimated stream of future earnings. Such calculations raise any number of practical questions concerning the estimates of future earning and the choice of an appropriate discount rate. Analysts advocate using long-term government bond rates, Treasury bill rates, or rates available on annuities (Edward, 1975; Harris, 1983; Carpenter et al., 1986). While the issue over which discount rate is most appropriate is important, this note addresses a more fundamental theoretical issue. We address the question of whether, in order to estimate the growth of future earnings and then to discount future earnings, it is necessary to forecast an inflation rate. In other words, should nominal earnings be discounted by (some appropriate) nominal rate of interest or can real earnings be discounted by a real rate of interest. Beginning with the neoclassical theory of the firm and intertemporal utility maximization theory we show that it is not necessary to forecast the rate of inflation; if discrete growth rates are combined with a discrete growth model or, if continuous growth rates are combined with a continuous growth model, the inflation rate cancels out. Thus, the nominal and real approach lead to precisely the same estimate of PDV.

While the above proposition has been denied in the literature (for a recent example, see Abraham, 1988), this note traces the reason for the denial to the application of growth rates derived for continuous time periods to discrete time growth models. Typically, textbooks in economics and finance express the relationship between the nominal rate of interest (observable market rates) and the real rate as:[1]

(1) $R = r + i$

*The authors are from the Department of Economics, Wake Forest University, Winston-Salem, NC.

[1] For examples from economics texts see Darby and Melvin, 1986, p. 82 and Hall and Taylor, 1988, p. 214. Examples from finance texts include Brigham, 1982, p. 148 and Schall and Haley, 1983, p. 37. By contrast, Barro, 1984, pp. 160-161; Hirshlifer, 1988, pp. 450-451 and Varian, 1987, pp. 188-189 all present careful treatments of real and nominal interest rates.

And, analysts often simply express the growth rate in earnings as:

(2) $G = g + i$

Where:
R = nominal interest rate
r = real interest rate
i = anticipated rate of inflation
G = growth rate of nominal earnings
g = growth rate of labor productivity

What is too rarely made clear is that this expression of Fisher's Equation and this expression of earnings growth rate are valid only at the limit of continuous compounding. When these expressions are applied to data defined for discrete periods (annual earnings, for example), using a discrete time growth model, differences do arise between the calculations of PDV of nominal earnings discounted by R and real earnings discounted by r.[2] But, the difference stems from a conceptual error, not because of some inherent superiority of nominal rates over real rates (or vice versa) as the growth and discount rates.

In Section II.A we derive the discrete growth rate for earnings for the neoclassical theory of the firm. In addition, the discount rate for discrete time is derived from intertemporal utility maximization theory. The two rates are combined with a discrete time growth model. In Section II.B the two rates are derived for continuous time. These two rates are combined with a continuous time growth model. In the models of Sections II.A and II.B it is seen that the inflation rate cancels out. In Section II.C we show that if the continuous growth rates are combined with a discrete growth model, (or vice versa) it would be erroneously concluded that the inflation rate does not cancel out and, therefore, must be forecasted. Section III contains the conclusions.

II. The Model

A. Discrete Time

Neoclassical economic theory holds that in competitive equilibrium, the nominal wage equals output price multiplied by the marginal product of labor.

(3) $W_t = P_t . MP_t$

Where:
W_t = nominal wage rate in period t
P_t = output price in period t
MP_t = marginal product of labor in period t

Growth in Earnings

The growth rate of the nominal annual wage is found by first expressing the annual change in the wage.

[2] Risk is assumed away for the purpose of this analysis.

(4) $\Delta W_t = [(P_{t-1} = \Delta P_t)(MP_{t-1} + \Delta MP_t)] - P_{t-1} \cdot MP_{t-1}$

Dividing equation 4 by W_{t-1}, yields:

(5) $$\frac{\Delta W_t}{W_{t-1}} = \frac{\Delta MP_t}{MP_{t-1}} + \frac{\Delta P_t}{P_{t-1}} + \frac{\Delta MP_t}{MP_{t-1}} \cdot \frac{\Delta P_t}{P_{t-1}}$$

Equation 5 can be rewritten as:

(6) $G = g + i + gi$

Where we measure G by $(\Delta W_t/W_{t-1})$, g by $(\Delta MP_t/MP_{t-1})$, and i by $(\Delta P_t/P_{t-1})$.[3] Given the neoclassical framework, equation 6 is the correct expression for *the growth rate* of nominal earnings over a discrete time period. The growth model for discrete time periods is:

(7) $W_t = W_{\emptyset}(1 + G)^t$

Growth in nominal earnings can be expressed, by substituting 6 into 7, as:

(8) $W_t = W_{\emptyset}[(1 + g)(1 + i)]^t$

Compare equations 6 and 2.

The Relationship of Nominal and Real Interest Rates

Intertemporal utility maximization theory implies that consumers choose a time path of real consumption such that:

(9) $$MRS_{C_t C_{t-1}} \equiv -\frac{\Delta C_t}{\Delta C_{t-1}} = 1 + r$$

Where:
C_t = real consumption in period t
MRS = marginal rate of substitution

The intertemporal equilibrium condition, in equation 9, states that the marginal rate of substitution between consumption (of real goods and services) next period and consumption this period equals one plus the real rate of interest. The same condition can be expressed in nominal terms, i.e., the $MRS_{E_t E_{t-1}}$ between money expenditures on consumption for different time periods.

[3] This assumes that the rate of wage inflation equals the general rate of inflation in the economy. The assumption is made, first, because the note focuses on the theoretical relationship of nominal and real interest rates and, second, because over time market forces will tend to adjust nominal wages by the general rate of inflation. Any deviations from the long-run rate of inflation in the rate of growth of nominal wages for a particular industry must be accounted for by industry specific factors. Moreover, as a practical matter the analyst is unlikely to ever have independent data on the *inflationary component* of raising nominal wages by industry. The assumption is widely made; witness industry COLA clauses, for example. On theoretical and practical grounds there seems to be no sound alternative to the assumption that over the long-run the average rate of wage inflation is equal to the general rate of inflation.

(10) $$MRS_{E_t E_{t-1}} \equiv -\frac{\Delta E_t}{\Delta E_{t-1}} = 1 + R$$

Where E_t = money expenditures for consumption in period t.

The relationship between nominal (money) and real goods is simply the price level. Equation 11 defines the relevant price levels.

(11) $$P_t \equiv \frac{\Delta E_t}{\Delta C_t} \text{ and } P_{t-1} \equiv \frac{\Delta E_{t-1}}{\Delta C_{t-1}}$$

The definition of inflation is given in equation 12:

(12) $$i = \frac{P_t - P_{t-1}}{P_{t-1}} \text{ or } 1 + i = \frac{P_t}{P_{t-1}}$$

Substituting from equations 9 through 12 in identity 13 yields,

(13) $$\frac{\Delta E_t}{\Delta E_{t-1}} \equiv \frac{\Delta E_t}{\Delta C_t} \cdot \frac{\Delta C_t}{\Delta C_{t-1}} \cdot \frac{\Delta C_{t-1}}{\Delta E_{t-1}}$$

or:

(14) $$1 + R = (1 + r)(1 + i) \text{ or } R = r + i + ri$$ [4]

Combining the discount rate of equation 14 with a discrete growth model such as equation 7 yields:

(15) $$PDVW_t = W_t \left[\frac{1}{(1+r)(1+i)}\right]^t$$

Where: $PDVW_t$ = present value of W_t

Substituting into 15 for W_t from 8 gives:

(16) $$PDVW_t = W_\emptyset \left[\frac{(1+q)(1+i)}{(1+r)(1+i)}\right]^t = W_\emptyset \left[\frac{1+q}{1+r}\right]^t$$

Equation 16 is the present value of the estimated amount of wages to be received in time period t. Equation 16 establishes the equivalence, in discrete time, of discounting nominal earnings with a nominal rate of interest and discounting real earnings with a real rate of interest.

To calculate the PDV of a stream of future earnings, where data are measured for discrete time periods, it is necessary to sum over equation 16.

[4] For an alternative derivation of equation 14 see Fama, 1975.

$$(17)\quad PDV = \sum_{t=\emptyset}^{T} W_{\emptyset} \left[\frac{1+G}{1+R}\right]^{t} = \sum_{t=\emptyset}^{T} W_{\emptyset} \left[\frac{(1+g)(1+i)}{(1+r)(1+i)}\right]^{t} = \sum_{t=\emptyset}^{T} W_{\emptyset} \left[\frac{1+g}{1+r}\right]^{t}$$

B. Continuous Time

The analysis of the last section is modified here to deal with continuous growth in earnings and continuous discounting of future earnings.

Growth of Earnings

The expression for the continuous growth rate is derived by differentiating equation 3 (above) with respect to time.

$$(18)\quad \frac{dW}{dt} = P\frac{dMP}{dt} + MP\frac{dP}{dt}$$

Dividing equation 18 through by W and using obvious notation yields:

$$(19)\quad G = g + i$$

Equation 19 is the expression for the growth rate of wages in continuous time.

In a continuous time growth modes (where $G = \frac{dW}{dt}\frac{1}{W}$), the rate of change of W can be written:

$$(20)\quad \frac{dW}{dt} = GW$$

The solution to his homogeneous, first order differential equation is:

$$(21)\quad W_t = W_{\emptyset}e^{Gt}$$

Combining the growth rate from equation 19 with the growth model from equation 21 yields 22.

$$(22)\quad W_t = W_{\emptyset}e^{(g+1)t}$$

Compare equations 22 and 8.

Nominal and Real Interest Rates: Continuous Compounding

If R is the *annual* rate of interest, a dollar invested at R will be worth 1 + R in one year. More generally a dollar will be worth $(1 + R/n)^n$ at the end of a year, where n is the frequency of compounding.

Generalizing equation 14 to introduce the frequency of compounding leads to

$$(23)\quad \left(1 + \frac{R}{n}\right)^{n} = \left(1 + \frac{r}{n}\right)^{n} \left(1 + \frac{i}{n}\right)^{n}$$

Using the result that as n goes to infinity,

(24) $$\lim_{n \to \infty} (1 + X/n)^n = e^X$$

implies that equation 23 can be rewritten as:

(25) $$e^R = e^r e^i$$

Taking the logarithms of both sides of equation 25, yields:

(26) $$R = r + i$$

Compare equations 26 and 14.

Combining the continuous discount rate from 26 with a continuous growth model, such as 21, yields:

(27) $$PDVW_t = W_t e^{-(r+i)t}$$

Substituting into 27 for Wt from 22 yields:

(28) $$PDVW_t = W_0 e^{(g+i)t} e^{-(r+i)t} = W_0 e^{(g-r)t}$$

Equation 28 is the present value of the estimated amount of wages to be received in time period t. Equation 28 establishes the equivalence, in continuous time, of discounting nominal earnings with a nominal rate of interest and discounting real earnings with a real rate of interest.

With continuous growth and discounting, the PDV of a stream of future earnings equals:

(29) $$PDV = \int_0^T W_0 e^{(g+i)t} e^{-(r+i)t} dt = \int_0^T W_0 e^{(g-r)t} dt$$

Integrating equation 29 yields:

(30) $$PDV = \frac{W_0}{r-g} \left(1 - e^{-(r-g)T} \right)$$

Compare equations 30 and 17.

C. Inconsistent Combinations

Combining the discrete growth rates (equations 6 and 14) with a discrete growth model (equation 7) yields equations 8 and 15 respectively. And, combining the continuous growth rates (equations 19 and 26) with a continuous growth model (equation 21) yields equations 22 and 27 respectively. The end results are equations 16 and 28 in which the inflation rate cancels. An inconsistent combination of discrete growth rates with a continuous growth model or continuous growth rates with a discrete growth model would, however, lead to the conclusion that the inflation rate does not cancel and thus that the inflation rate must be forecast.

Consider the combination of equations 19 and 26, continuous growth rates, with a discrete growth model equation 7. The result is:

$$(31)\quad PDVW_t = W_0\left(\frac{1+q+i}{1+r+i}\right)^t$$

Also, consider the combination of discrete growth rates, equations 6 and 14, with a continuous growth model, equation 21. The result is:

$$(32)\quad PDVW_t = W_0 e^{(g+i+gi)t} e^{-(r+i+ri)t}$$

One would conclude from equation 31 or equation 32 that the inflation rate does not cancel. However, we have shown that equations 31 and 32 are the result of a conceptual error.

III. Conclusion

We began with the neoclassical theory of the firm and the theory of intertemporal utility maximization. From the neoclassical theory, the discrete time and continuous time growth rates for earnings were derived. From the theory of intertemporal utility maximization the discrete time and continuous time interest rates were derived. When the discrete time earnings growth rate and the discrete time interest rate were combined with a discrete time growth model in order to calculate the PDV of estimated future earnings it was seen that the inflation rate canceled. Similarly, when the continuous time earnings growth rate and discount rate were combined with a continuous time growth rate, the inflation rate cancelled. These results obviate the need to forecast the inflation rate when calculating the PDV of an estimate of future earnings. We also showed that if one makes the conceptual error of combining the continuous time earnings growth rate and discount rate with a discrete growth model, or combining the discrete time earnings growth rate and discount rate with a continuous growth model, the inflation rate would not cancel. Here the analyst can let the available data dictate whether to express the calculations in nominal or real terms without (the unfounded) fear of biasing the results.

There are dozens of other useful implications to this analysis. To mention but one (leaving the reader to discover others), statistically unbiased estimates of the average annual rate of inflation for various future time periods can be obtained by substituting into equation 14 the nominal rate of interest on government bonds (of the maturity consistent with the desired forecast period, e.g., 3 year government notes for a 3 year estimate of inflation; 20 year government bonds for a 20 year forecast) and the historic real rate of interest. Then simply solve the equation for the unbiased estimate of the annual rate of inflation for the relevant period.

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